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Higher-dimensional Formulation of Counterterms

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ABSTRACT

It is by now well established that divergences of the on-shell action for asymptotically AdS solutions can be cancelled by adding covariant local boundary counterterms to the action. Here we show that although one can still renormalise the action for asymptotically $AdS_p \times S^q$ solutions using local boundary counterterms the counterterm action is not covariant since the conformal boundary is degenerate. Any given counterterm action is defined with respect to specific coordinate frame and gauge choices.

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Contents

1 Introduction

The correspondence between supergravity (string) theories in AdS backgrounds and their dual conformal field theories has by now passed many tests and produced a number of interesting insights into strong coupling behaviour of field theory [1], [2], [3], [4]. It has also provided a partial solution to a long standing problem in relativity: how to define finite masses and charges for a given asymptotically AdS metric without resorting to an ill-defined background subtraction procedure.

The method of holographic renormalisation has been developed systematically for asymptotically AdS solutions [5], [6], [7], [8], [9], [10], [11], [12]. A finite renormalised action can be constructed as a functional of boundary data for the bulk fields which are sources of operators in the dual field theory. One can obtain from this action the renormalised correlation functions by functional differentiation.

AdS backgrounds in string theory appear as backgrounds of the form $AdS_p \times X^q$ where X^q is an Einstein manifold of positive curvature whose symmetries relate to the R symmetries of the dual theory. So far all holographic renormalisation has been carried out at the level of the p-dimensional gauged supergravity action obtained by dimensionally reducing on X^q.

Here we attempt to extend the program of holographic renormalisation to the $(p + q)$-dimensional action. There are a number of motivations for this. It would be interesting to understand how the full higher dimensional spacetime is reconstructed from field theory data. Also the Kaluza-Klein reduction is complex and explicitly matching higher to lower dimensional fields is not easy even for quite simple solutions, such as BPS brane distributions. There are a number of solutions which are known in either higher or lower dimension but which have not been explicitly lifted or reduced. One example is the GPPZ flow in five dimensions [13] which should lift to a Polchinski-Strassler solution [14]. Part of the lift was carried out in [15] but the full ten-dimensional set of fields is still not known. So it would be useful, purely as a calculational tool, to be able to calculate the renormalised action, mass and so on in both dimensions.

One should be able to extend the techniques of holographic renormalisation to supergravity in any background with a dual field theory description for which UV divergences can be regulated and removed by a suitable renormalisation scheme. In particular, solutions of gauged supergravity theories such as those considered in [16] should admit holographic renormalisation, even though they are not asymptotically AdS. Such backgrounds share a significant feature with $AdS_p \times X^q$ backgrounds: the conformal boundaries are degenerate. In the latter case, this degeneracy is manifest since the boundary is the product of some $(p-1)$-dimensional manifold and (the collapse of X^q to) a point.

Degenerate boundaries were explored in [17] and further understanding the implications of such degeneracy is the main topic of this paper. The main result is the following. Although we can renormalise the action using a local boundary counterterm action, the counterterms cannot be written in terms of covariant quantities intrinsic to the boundary. This means that for a given counterterm action we will also have to specify particular coordinate and gauge choices. We should emphasise that nonetheless we can still renormalise the action in a well-defined and systematic procedure.

To demonstrate these points we will first consider divergences and counterterms for asymptotically $AdS_3 \times S^3$ spacetimes. In §3 we then discuss the physically more interesting case of asymptotically $AdS_5 \times S^5$ spacetimes and demonstrate explicitly how the renormalisation procedure works from the higher-dimensional perspective.

2 Divergences and counterterms for $AdS_3 \times S^3$ spacetimes

Three-dimensional counterterms have been less studied than those in higher dimensions. Although they are included in the analysis of [8], [9] there is no canonical three-dimensional gauged supergravity (and hence dual conformal field theory) which has been studied. Ultimately this is because there are few known consistent truncations of spherical compactifications down to three dimensions which also admit anti-de Sitter metrics as solutions [18].

For example, if one starts from the common bosonic sector of ten-dimensional supergravity theories and compactifies on a seven-sphere, it is consistent to truncate to just the massless modes but the resulting gauged supergravity theory admits domain wall rather than anti-de Sitter solutions [18]. Other consistent spherical compactifications involving (for simplicity) only a small set of higher-dimensional fields are extremely rare. In fact, the only other example which has been discussed explicitly is the compactification of the six-dimensional bosonic string low-energy effective Lagrangian on a three-sphere to three dimensions [18].

Working at the purely classical level we restrict to the simple Lagrangian

$$\mathcal{L}_6 = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{12}e^{-a\phi}(F_3)^2, \tag{2.1}$$

where F_3 is the Kalb-Ramond three-form and $a = \sqrt{2}$. The value of the constant a is critical in ensuring that the spherical compactification can be truncated consistently just to massless modes [18]. Now the conformal anomaly term adds a term of the form

$$\delta\mathcal{L} = -\tfrac{1}{2}q^2 e^{\frac{1}{2}a\phi} \tag{2.2}$$

to the Lagrangian. This extra contribution is a cosmological term; the spherical compactification remains consistent with this term. However if we include this term $AdS_3 \times S^3$ is not a solution of the equations of motion; we have to switch on the six-dimensional dilaton and consider domain wall solutions in three dimensions instead. Since here we are working purely at the classical level, and do not need to appeal to any microscopic description, there is no objection to omitting the conformal anomaly term and working purely with (2.1).

The six-dimensional equations of motion are then

$$\begin{aligned} R_{mn} &= \tfrac{1}{2}\partial_m\phi\partial_n\phi + \tfrac{1}{4}e^{-\sqrt{2}\phi}F_{mpq}F_n{}^{pq} - \tfrac{1}{24}e^{-\sqrt{2}\phi}F_{pqr}F^{pqr}g_{mn}; \\ D_m\partial^m\phi &= -\tfrac{\sqrt{2}}{12}e^{-\sqrt{2}\phi}F_{pqr}F^{pqr}; \\ d(e^{-\sqrt{2}\phi} * F_3) &= 0. \end{aligned} \tag{2.3}$$

Here we work in Lorentzian signature and the index m, n runs between 0 and 5. These equations manifestly admit a solution with constant dilaton and self-dual three-form such that

$$\begin{aligned} ds_6^2 &= ds_3^2 + l^{-2}d\Omega_3^2; \\ F_3 &= 2l\epsilon_3 + 2l^{-2}d\Omega_3, \end{aligned} \tag{2.4}$$

where $d\Omega_3^2$ is the unit metric on the three-sphere and ds_3^2 is an Einstein metric of constant negative curvature satisfying $R_{\mu\nu} = -2l^2 g_{\mu\nu}$.

We use the following Kaluza-Klein ansatz for the spherical reduction

$$\begin{aligned} ds_6^2 &= Y^{\frac{1}{4}}\left(\Delta^{\frac{1}{2}}ds_3^2 + l^{-2}\Delta^{-\frac{1}{2}}T_{ij}^{-1}D\mu^i D\mu^j\right); \\ e^{\sqrt{2}\phi} &= \Delta^{-1}Y^{\frac{1}{2}}; \\ e^{-\sqrt{2}\phi} * F_3 &= -lU\epsilon_3 + l^{-1}T_{ij}^{-1} * DT_{jk} \wedge (\mu^k D\mu^i) - \tfrac{1}{2}l^{-2}T_{ik}^{-1}T_{jl}^{-1} * F^{ij} \wedge D\mu^k \wedge D\mu^l \\ &\quad + \tfrac{m}{6}l^{-2}\epsilon_{ijkl}\mu^i D\mu^j \wedge D\mu^k \wedge D\mu^l; \\ F_3 &= mY\epsilon_3 + \tfrac{1}{6}\epsilon_{ijkl}\Big(l^{-2}U\Delta^{-2}D\mu^i \wedge D\mu^j \wedge D\mu^k\mu^l - 3l^{-1}\Delta^{-1}F^{ij} \wedge D\mu^k T_{li'}\mu^{i'} \\ &\quad - 3l^{-2}\Delta^{-2}D\mu^i \wedge D\mu^j \wedge DT_{ki'}T_{lj'}\mu^{i'}\mu^{j'}\Big), \end{aligned} \tag{2.5}$$

with the following definitions

$$\mu^i\mu^i = 1; \qquad \Delta = T_{ij}\mu^i\mu^j;$$

$$
\begin{aligned}
U = 2T_{ik}T_{jk}\mu^i\mu^j - \Delta T_{ii}; \qquad & Y = \det(T_{ij}); \\
D\mu^i &= d\mu^i + lA^{ij}\mu^j; \\
DT_{ij} &= dT_{ij} + lA^{ik}T_{kj} + lA^{jk}T_{ik}; \\
F^{ij} &= dA^{ij} + lA^{ik}\wedge A^{kj}.
\end{aligned} \tag{2.6}
$$

In addition, ϵ_3 in (2.5) is the volume form of the three-dimensional metric. The indices i, j are $SO(4)$ indices and hence run between 1 and 4. The three-sphere is parametrised in this ansatz by a set of direction cosines. The effective three-dimensional Lagrangian is then

$$
\begin{aligned}
\mathcal{L}_3 &= R_3 - \tfrac{1}{16}Y^{-2}(\partial Y)^2 - \tfrac{1}{4}\tilde{T}_{ij}^{-1}(D\tilde{T}_{jk})\tilde{T}_{kl}^{-1}(D\tilde{T}_{li}) - \tfrac{1}{2}m^2 Y \\
&\quad -\tfrac{1}{8}Y^{-\frac{1}{2}}\tilde{T}_{ik}^{-1}\tilde{T}_{jl}^{-1}F^{ij}F^{kl} - \tfrac{1}{2}l^2 Y^{\frac{1}{2}}(2\tilde{T}_{ij}\tilde{T}_{ij} - \tilde{T}_{ii}^2),
\end{aligned} \tag{2.7}
$$

where we have found it convenient to introduce the unimodular $\tilde{T}$ where

$$
T_{ij} = Y^{\frac{1}{4}}\tilde{T}_{ij}. \tag{2.8}
$$

Note that although in toroidal compactifications we can dualise the three-dimensional gauge potentials to scalars this is not possible for the gauged theory. The resulting three-dimensional equations of motion are

$$
\begin{aligned}
R_{\mu\nu} &= \tfrac{1}{16}Y^{-2}(\partial_\mu Y)(\partial_\nu Y) + \tfrac{1}{4}\tilde{T}_{ij}^{-1}(D_\mu\tilde{T}_{jk})\tilde{T}_{kl}^{-1}(D_\nu\tilde{T}_{li}) + \tfrac{1}{2}m^2 Y g_{\mu\nu} \\
&\quad +\tfrac{1}{2}l^2 Y^{\frac{1}{2}}(2\tilde{T}_{ij}\tilde{T}_{ij} - \tilde{T}_{ii}^2)g_{\mu\nu} + \tfrac{1}{4}Y^{-\frac{1}{2}}\tilde{T}_{ik}^{-1}\tilde{T}_{jl}^{-1}F^{ij}_{\mu\rho}F^{kl\rho}_{\nu}; \\
D_\mu(Y^{-2}\partial^\mu Y) &= -\tfrac{1}{2}Y^{-\frac{3}{2}}\tilde{T}_{ik}^{-1}\tilde{T}_{jl}^{-1}F^{ij}_{\mu\nu}F^{kl\mu\nu} + 2l^2Y^{-\frac{1}{2}}(2\tilde{T}_{ij}\tilde{T}_{ij} - \tilde{T}_{ii}^2) + 4m^2; \\
D_\mu(\tilde{T}_{ik}^{-1}D^\mu\tilde{T}_{kj}) &= 2l^2Y^{\frac{1}{2}}(2\tilde{T}_{ik}\tilde{T}_{jk} - \tilde{T}_{ij}\tilde{T}_{kk}) - Y^{-\frac{1}{2}}\tilde{T}_{lm}^{-1}\tilde{T}_{ik}^{-1}(F^{lk}_{\mu\nu}F^{mj\mu\nu}) \\
&\quad -\tfrac{1}{4}\delta_{ij}\left(2l^2Y^{\frac{1}{2}}(2\tilde{T}_{lk}\tilde{T}_{lk} - (\tilde{T}_{kk})^2) - Y^{-\frac{1}{2}}\tilde{T}_{lm}^{-1}\tilde{T}_{nk}^{-1}F^{lk}_{\mu\nu}F^{mn\mu\nu}\right); \\
D(Y^{-\frac{1}{2}}\tilde{T}_{ik}^{-1}\tilde{T}_{jl}^{-1} * F^{kl}) &= -2l\tilde{T}_{k[i}^{-1} * D\tilde{T}_{j]k} - \tfrac{1}{2}m\epsilon_{ijkl}F^{kl}.
\end{aligned} \tag{2.9}
$$

Here each derivative D is of the appropriate fully covariantised form. The (locally) AdS_3 solutions of these equations are recovered by setting

$$
m^2 = 4l^2; \qquad \tilde{T}_{ij} = \delta_{ij}; \qquad Y = 1; \qquad F^{ij} = 0. \tag{2.10}
$$

We would like to determine the IR divergences of the action in both six and three dimensions. Let us consider first the three-dimensional case. To evaluate the on-shell divergences we need to solve the field equations in the vicinity of the AdS boundary to sufficient order to determine all IR divergences. To do this we use the by now well known results of Fefferman and Graham [19], [20] that any $(d+1)$-dimensional metric of negative curvature admits an asymptotic expansion near the boundary of the form

$$
ds^2_{(d+1)} = \frac{d\rho^2}{l^2\rho^2} + \frac{1}{l^2\rho^2}g_{\alpha\beta}dx^\alpha dx^\beta, \tag{2.11}
$$

where the d-dimensional metric g admits an expansion

$$g = g^0 + \rho^2 g^2 + ... \rho^d g^d + h^d \rho^d \ln \rho + ... \tag{2.12}$$

The logarithmic term appears only when d is even and only even powers of ρ appear up to this order. l is the scale parameter which appeared above. We will also need to expand the matter fields about their background (or asymptotic) values given in (2.10) and then solve the full set of coupled equations. This turns out to be much easier than one might have naively expected. The key simplification is that both the vectors and all the scalars have to admit expansions that start at powers of ρ too high to contribute to the field equations or to the action, to the order necessary to determine all divergences. The chain of arguments required to determine this is quite complicated and involves all the field equations.

Firstly let us expand the two dimensional metric from (2.11) as

$$g_{\alpha\beta} = g^0_{\alpha\beta} + \rho^2 g^2_{\alpha\beta} + \rho^2 \sum_{p=1} h^p_{\alpha\beta} \ln \rho + ..., \tag{2.13}$$

where the ellipses denote terms of higher order where the expansion breaks down. Although for purely gravitational divergences we need only include the first logarithmic term $h^1_{\alpha\beta}$, it is known that with matter source terms one can have further logarithmic terms. Explicitly calculating the curvature of this metric we find the following

$$\begin{aligned} R_{\rho\rho} &= -\frac{2}{\rho^2} - l^2 \left(\mathrm{tr}((g^0)^{-1}h^1) - \mathrm{tr}((g^0)^{-1}h^2) - 2\ln\rho\,\mathrm{tr}((g^0)^{-1}h^2) \right. \\ &\quad \left. -3\ln\rho(1+\ln\rho)\mathrm{tr}((g^0)^{-1}h^3) \right) + ...; \\ (g^0)^{\alpha\beta} R_{\alpha\beta} &= -\frac{4}{\rho^2} + l^2 R^0 - l^2 \left(\mathrm{tr}((g^0)^{-1}h^2) - 3\ln\rho\,\mathrm{tr}((g^0)^{-1}h^3) \right) + ... \end{aligned} \tag{2.14}$$

where we have retained only terms up to order h^3 and R^0 is the curvature of the metric g^0. All contractions are taken in the metric g^0.

Now using the Einstein equation in (2.9) we can deduce that to preserve the requisite form of the metric (2.10) the matter fields must satisfy

$$\begin{aligned} \delta Y &= \rho^2 \sum_{p=0} Y^{(p)}(x^\alpha)(\ln\rho)^p + .. \\ \delta \tilde{T}_{ij} &= \rho \sum_{p=0} t^{(p)}_{ij} (\ln\rho)^p + \rho^2 \sum_{p=0} \tau^{(p)}_{ij} (\ln\rho)^p + ...; \\ F^{ij}_{\mu\nu} F^{\mu\nu}_{ij} &\leq O(\rho^2), \end{aligned} \tag{2.15}$$

where we will give a more detailed expansion of the vector fields later. The first two expansions are perturbations about the background values given in (2.10). Since $\tilde{T}$ is unimodular, expanding out the determinant we find that each $t^{(p)}_{ij}$ must be traceless whilst

$$\tau^{(r)}_{ii} = \tfrac{1}{2} \sum_{p+q=r} t^{(p)}_{ij} t^{(q)}_{ji}. \tag{2.16}$$

From these constraints we find that

$$(2\tilde{T}_{ij}\tilde{T}_{ij} - \tilde{T}^2_{ii}) = -8 + O(\rho^3), \tag{2.17}$$

and it is this which, using the equation for Y, forces the perturbation in Y to be at least as small as ρ^2. This follows from writing the equation as

$$D_\mu(Y^{-2}\partial^\mu Y) = 8l^2\delta Y + O(\rho^2), \tag{2.18}$$

from which it is easy to show that δY cannot be of order ρ. Given that δY is of order ρ^2 or smaller, it turns out that it will not contribute to the Einstein or other field equations to the required order and can henceforth be neglected.

A similar, though slightly more subtle, argument can now be used to show that $t^{(p)}_{ij} = 0$. Expanding out the equation for $\tilde{T}_{ij}$, we find that $t^{(p)}_{ij}$ must satisfy

$$\sum_{p=0} D_\rho D^\rho(\rho t^{(p)}_{ij}(\ln\rho)^p) \leq O(\rho^2). \tag{2.19}$$

This can evidently not be satisfied for non-zero t_{ij} when the gauge field vanishes. It can also not be satisfied even when the gauge field is switched on. To demonstrate this, let us retain only the t^0_{ij} term for simplicity. Using the Einstein equations to constrain the gauge field strengths to be at least as small as in (2.15) the gauge potentials should be expanded as

$$\begin{aligned} A^{ij}_\rho &= \rho^{-1}\sum_{p=0}\beta^{ij(p)}(x^\alpha)(\ln\rho)^p; \\ A^{ij}_\alpha &= \sum_{p=0}\alpha^{ij(p)}_\alpha(x^\alpha)(\ln\rho)^p. \end{aligned} \tag{2.20}$$

Note that the α components must be of this order to ensure the absence of ρ^{-2} terms in the field strength. Substituting the gauge potential into the equation (2.19), and for simplicity of the argument retaining only the $p = 0$ terms in the potentials, we find the following constraint

$$\begin{aligned} v^{ij} &\equiv t^{ij} + \beta^{ik}t^{kj} + \beta^{jk}t^{ki}; \\ v^{ij} &= \beta^{ik}v^{kj} + \beta^{jk}v^{ki}, \end{aligned} \tag{2.21}$$

where we have suppressed p labels. v^{ij} is a traceless symmetric matrix, like t^{ij}. These constraints can evidently not be satisfied for non-zero v^{ij} or t^{ij} since, for example, the second constraint implies

$$\det\beta = \det(1+\beta). \tag{2.22}$$

It is straightforward to generalise these arguments to include non-zero values of p. Thus we have proved that the expansion of $\tilde{T}$ starts at order ρ^2 and to the required order the three-dimensional field equations reduce to

$$\begin{aligned}
R_{\rho\rho} &= -2l^2\rho^{-2} - \tfrac{1}{4}l^2\rho^2(g^0)^{\alpha\beta}F^{ij}_{\rho\alpha}F^{ij}_{\rho\beta} + ...; \\
(g^0)^{\alpha\beta}R_{\alpha\beta} &= -2l^2\rho^{-2}(g^0)^{\alpha\beta}g_{\alpha\beta} - \tfrac{1}{4}l^2\rho^2(g^0)^{\alpha\beta}F^{ij}_{\rho\alpha}F^{ij}_{\rho\beta} + ...; \\
D(*_3F^{kl}) &= -l\epsilon_{ijkl}F^{kl} + ...
\end{aligned} \tag{2.23}$$

For the gauge fields we have used the fact that, following from (2.20), the field strengths can be expanded as

$$\begin{aligned}
F^{ij}_{\alpha\beta} &= \sum_{p=0} f^{ij(p)}_{\alpha\beta}(ln\rho)^p + ...; \\
F^{ij}_{\rho\alpha} &= \rho^{-1}\sum_{p=0} g^{ij(p)}_{\rho\alpha}(\ln\rho)^p + ...,
\end{aligned} \tag{2.24}$$

and hence the components $F^{ij}_{\alpha\beta}$ will not contribute to the required order.

Solving the last of the equations in (2.23) would be complex but this turns out to be unnecessary since the two Einstein equations force

$$\begin{aligned}
\mathrm{tr}((g^0)^{-1}h^p) &= 0 \quad \forall p; \\
F^{ij}_{\alpha\rho}F^{ij}_{\alpha\rho} &< O(\rho^{-2}); \\
\mathrm{tr}((g^0)^{-1}g^2) &= -\tfrac{1}{2}l^2R^0,
\end{aligned} \tag{2.25}$$

so that, as claimed above, the only IR divergences of the action are the gravitational ones. It is important in what follows that we have excluded ρ corrections to any of the fields.

The three-dimensional action is

$$\mathcal{S} = \frac{1}{2\kappa^2}\int\sqrt{g}\mathcal{L}_3 - \frac{1}{\kappa^2}d^2x\int K\sqrt{h}, \tag{2.26}$$

where as usual the second term is the Gibbons-Hawking boundary term with K the trace of the second fundamental form of the boundary. Explicitly calculating the IR divergences by cutting off the boundary at $\rho = \epsilon$, we find

$$\mathcal{S}_{\mathrm{div}} = -\frac{1}{\kappa^2}\int d^2x\sqrt{g^0}\left(\frac{1}{l\epsilon^2} - (R^0 l)\ln\epsilon\right), \tag{2.27}$$

which leads to the following counterterm action

$$\mathcal{S}_{\rm cov} = \frac{1}{\kappa^2}\int d^2x\sqrt{h}\left(l^{-1} - lR[h]\ln\epsilon\right), \tag{2.28}$$

in terms of the induced metric on the boundary h and its curvature $R[h]$. This completes the calculation of divergences and counterterms in three dimensions. Although the final answer was very simple, involving only the induced metric on the boundary and its curvature, we solved the full set of equations in order to find all possible divergences and explicitly showed that, for example, there can be no terms in odd powers of ρ in the metric. This is significant in what follows.

Now let us consider the divergences and counterterms from the higher dimensional perspective. At first sight it seems as though the uplift to six dimensions will be trivial since all matter field perturbations are subleading. However, although we can and will set the gauge fields and Y perturbations to zero, we should retain the perturbation

$$\tilde{T}_{ij} = \delta_{ij} + \rho^2\tau_{ij}(x^\alpha), \tag{2.29}$$

where τ_{ij} is a traceless symmetric matrix. Although this perturbation does not contribute to the IR divergences it deforms the three sphere when we uplift the solution to six dimensions and is the leading order field perturbation.

Using the Kaluza-Klein ansatz of (2.5) we can now lift our three dimensional fields up to six dimensions. This gives

$$\begin{aligned}
ds_6^2 &= (1+\tfrac{1}{2}\rho^2 G)ds_3^2 + l^{-2}(1-\tfrac{1}{2}\rho^2 G)\left(g^0_{ab} + \rho^2 g^2_{ab} de^a de^b\right) + ...; \\
g^2_{ab} &= -Gg^0_{ab} - \tfrac{1}{2}G_{;(ab)}; \\
G_{;a}{}^{a} &= -8G \quad \rightarrow \quad \mathrm{tr}((g^0)^{-1}g^2) = G; \\
e^{\sqrt{2}\phi} &= (1-\rho^2 G + ...); \\
F_3 &= 2l\epsilon_3 + 2l^{-2}(1-2\rho^2 G)\eta^0_3 - 4l^{-2}d(\rho^2 G)\wedge\partial\eta^0_3 ...,
\end{aligned} \tag{2.30}$$

where in the first line ds_3^2 is the three-dimensional metric already determined and g^0_{ab} is an Einstein metric on the unit three-sphere, with coordinates x^a where $a = 1, 3$. The deformation of the spherical metric is defined in the second line, using the covariant derivative on the three-sphere and a function G which is related to the traceless symmetric tensor τ_{ij} already defined as

$$G(x^\beta; x^a) = \tau_{ij}(x^\beta)\mu^i\mu^j, \tag{2.31}$$

and is thus an $l = 2$ harmonic on the three-sphere. This is the implication of the third line of (2.30). In the last line, ϵ_3 is the volume form of the full three-dimensional metric whilst

η_3^0 is the volume form of the unit three-sphere. $\partial\eta_3^0$ is the interior derivative of the volume form; this final term in F_3 ensures that it is closed.

With this expansion of the six-dimensional fields we can proceed to calculating the divergences and counterterms. We could of course have worked out the expansions of the fields by using an appropriate ansatz to solve the six-dimensional equations directly. There are a number of reasons for not doing this. Firstly, it is quite difficult to find an appropriate ansatz that is sufficiently general. Secondly, even given an appropriate ansatz, the six-dimensional field equations are actually slightly more involved than the dimensional reduced equations, essentially because of the explicit spherical dependence. Thirdly, we do of course already know the three-dimensional equations explicitly whereas for any six-dimensional ansatz we would need to work them out!

Finally, and most importantly, the explicit frame dependence of the six-dimensional counterterm action and its relationship to the covariant counterterm action in three dimensions is much clearer when we follow the route of uplifting from three dimensions. Lack of covariance is the crucial issue and it is important to identify clearly the origins of the problem.

The observant reader would be right to have misgivings at this stage, though, for the following reason. Although all solutions of the three-dimensional field equations are, by construction, solutions of the six-dimensional equations, the reserve is certainly not true. Any six-dimensional solution which involves three-dimensional fields not included in the supergravity multiplet (i.e. massive Kaluza-Klein excitations) will not solve the three-dimensional field equations used here and will hence be excluded from our analysis. We are not ignoring this point and will shortly return to discuss it in detail.

For now let us proceed with the uplift of the three-dimensional solutions. One would expect that the six-dimensional action should be taken to be, as usual,

$$\mathcal{S}_6 = \frac{1}{2\kappa_6^2}\int d^6x\sqrt{g}\mathcal{L}_6 - \frac{1}{\kappa_6^2}\int d^5xK\sqrt{h}, \tag{2.32}$$

where we have used the six-dimensional coupling constant and included the Gibbons-Hawking boundary term. However, although this action can be shown to reproduce the same field equations as the three-dimensional action when one substitutes the Kaluza-Klein ansatz, it does not reproduce the same action. The key to the discrepancy is the bulk term: the on-shell value is

$$\mathcal{S}_{\rm bulk} = -\frac{1}{24\kappa_6^2}\int d^6x\sqrt{g}e^{-\sqrt{2}\phi}(F_3)^2. \tag{2.33}$$

However, to leading order F_3 is self-dual and the total contribution of the bulk term to the divergences is

$$\mathcal{S}_{\rm bulk} = -\frac{1}{\kappa_6^2 l^4} \ln \epsilon \int d^2x \sqrt{g^0} \int d^3x \sqrt{\eta^0} G(x^\alpha; x^a) = 0, \tag{2.34}$$

where η^0 is the measure on the three-sphere. Since G is an $l = 2$ harmonic, the integral of G over the three-sphere vanishes and hence there are no divergent contributions from the bulk term. The divergence from the Gibbons-Hawking boundary term is

$$\mathcal{S}_{\rm GH} = -\frac{2}{\kappa_6^2 l^4 \epsilon^2} \int d^2x \sqrt{g^0} \int d^3x \sqrt{\eta^0} = -\frac{4\pi^2}{\kappa_6^2 l^4 \epsilon^2} \int d^2x \sqrt{g^0}. \tag{2.35}$$

Thus the divergent part of the action does not agree with the three-dimensional action! There is no reason why it should since so far we have demanded only that the field equations are equivalent for a consistent reduction, not that the actions are equivalent. Nonetheless there are other boundary terms that one could add to the six-dimensional action (or indeed the three-dimensional action) without affecting the equations of motion. The term that is relevant here is

$$\delta \mathcal{S} = \frac{1}{12\kappa_6^2} \int d\Sigma^\sigma A^{\mu\nu} F_{\mu\nu\sigma} e^{-\sqrt{2}\phi}. \tag{2.36}$$

Note that although this term lives on the boundary it is *not* a counterterm because it does not just involve quantities intrinsic to the boundary. It depends on the embedding of the boundary hypersurface into the bulk.

There is no reason *a priori* why we need to demand equivalence between higher and lower dimensional actions. We could proceed with renormalising the higher dimensional action with no such boundary term: the renormalised action would differ from that for the same solution evaluated with three dimensional fields, but the renormalisation procedure would of course still be consistent. We take the view here that it is more convenient for the higher and lower dimensional actions to be equivalent.

The physical interpretation of adding such a term to the action is that we are shifting between different thermodynamic ensembles, from one where the total electric charge is fixed to one where it becomes a thermodynamic variable. The addition of such a term is well-known in the context of, for example, calculating the free energy for black hole metrics [21]. Let us illustrate this in the best known case, namely four-dimensional Reissner-Nordstrom black holes. It is straightforward to show by explicit calculation that the action given by

$$\mathcal{S} = \frac{1}{2\kappa_4^2} \int d^4x \left(R - \frac{1}{4} F^2 \right) - \frac{1}{\kappa_4^2} \int d^3x K \sqrt{h}, \tag{2.37}$$

when evaluated on-shell for Reissner-Nordstrom electric black holes (using the usual background subtraction methods) is equal to the free energy in a grand canonical ensemble such that

$$\mathcal{S} = -\beta F = \beta M - \beta Q \Phi_h - S_h, \tag{2.38}$$

where F is the free energy, β is the inverse temperature, M is the mass, Q is the electric charge, Φ_h is the difference between the electric potential at the horizon and infinity and S_h is the entropy (all in appropriate units). If we wish to calculate the free energy in a canonical ensemble in which the electric charge is fixed, then we have to add an additional term to the action (2.37) of the form

$$\delta\mathcal{S} = \frac{1}{4\kappa_4^2} \int d\Sigma^\sigma A^\mu F_{\sigma\mu}, \tag{2.39}$$

so that

$$\mathcal{S} + \delta\mathcal{S} = \beta M - S_h. \tag{2.40}$$

It turns out that to ensure the divergent part of the on-shell action is the same in six dimensions as in three we need to work in the canonical ensemble where the electric charge is fixed. This is an interesting point which had not been noticed before and may be relevant in considering stability of anti-de Sitter black holes from a higher-dimensional perspective, where the choice of ensemble is quite subtle [22], [23].

One could in principle show, using the Kaluza-Klein ansatz, that the six-dimensional action with this boundary term reproduces the three-dimensional action. This would require us showing that the higher and lower dimensional sets of Einstein equations are equivalent, which was never explicitly proved for this reduction [18]. We checked this equivalence for a truncated set of three-dimensional fields, the metric and a single active scalar, and the corresponding uplift; this is an adequate check provided that, as expected, the Einstein equations are also equivalent.

With the addition of the boundary term, the total divergences of the six dimensional action are

$$\mathcal{S}_{\rm div} = -\frac{1}{\kappa_6^2 l^4} \int d^2x \sqrt{g^0} \int d^3x \sqrt{\eta^0} \left(\epsilon^{-2} - l^2 R^0 \ln \epsilon\right), \tag{2.41}$$

which evidently agrees with the three-dimensional divergences since $\kappa_6^2 = 2\pi^2 l^{-3} \kappa^2$. The logarithmic term originates in the form of the two-form potential expanded near the boundary

$$A_{\alpha\beta} = -l^{-2}\rho^{-2} - \tfrac{1}{2} R^0 \ln \rho + ... \tag{2.42}$$

where we have chosen the gauge such that $A_{\rho\alpha} = 0$.

To render the action finite we would now like to define an appropriate counterterm action to cancel the divergences. The counterterm action must not affect the equations of motion and must hence be defined entirely in terms of quantities intrinsic to the regularising boundary. Furthermore, we would hope that the action can be written in terms of covariant quantities on the boundary, such as the curvature of the induced metric. Here however we run into a fundamental problem: the boundary is degenerate since the three sphere remains of finite size as we take the IR cutoff to infinity. The implication of this is that although we can always subtract off the divergences by defining an appropriate counterterm action this action cannot in general be written in terms of covariant quantities of the boundary fields. Since the boundary is degenerate there is no concept of five-dimensional covariance.

Holography for degenerate boundary metrics was discussed in [17] and we will now review some of the arguments that appeared there. Let us first consider an Einstein metric of negative curvature that admits an expansion of the Fefferman-Graham type (2.11) and (2.12). Then the induced metric on the boundary is

$$h_{\alpha\beta} = l^{-2}\rho^{-2} g_{\alpha\beta}, \tag{2.43}$$

where g is defined in (2.12). The divergences of the action will be expressible as a power series

$$\mathcal{S}_{\text{div}} = \int d^d x \sqrt{g^0} \left(a_d(x^\alpha)\epsilon^{-d} + ...A(x^\alpha) \ln \epsilon + ... \right). \tag{2.44}$$

A covariant counterterm action can be constructed using the following procedure. Each time we take a derivative in the boundary we add a power of ρ. Thus the measure is of order ρ^{-d}, the Ricci curvature is of order ρ^2, the Ricci tensor squared is of order ρ^4 and so on. To cancel the divergences we can therefore construct an action of the form

$$\mathcal{S}_{\text{ct}} = \int d^d x \sqrt{h} \left(\alpha_0 + \alpha_2 R[h] + \alpha_4 (R[h]^2 + \beta R_{\alpha\beta}[h] R^{\alpha\beta}[h]) + ... \right), \tag{2.45}$$

where we have omitted logarithmic terms and the α_i are constants. Then the first term cancels the ϵ^{-d} divergence whilst the term in $R[h]$ is needed to cancel the ϵ^{-d+2} divergence and so on.

This procedure breaks down, however, for degenerate metrics. Suppose that the boundary metric is of the form considered here, namely,

$$h_{ef} = \begin{pmatrix} l^{-2}\epsilon^{-2}(g^0_{\alpha\beta} + \epsilon^2 g^2_{\alpha\beta} + ..) & 0 \\ 0 & l^{-2}(g^0_{ab} + \epsilon^2 g^2_{ab} + ...) \end{pmatrix} \tag{2.46}$$

so that the measure is of order ϵ^{-2}. The Ricci curvature of this metric is

$$R_{ef} = \begin{pmatrix} R^0_{\alpha\beta} & 0 \\ 0 & (2g^0_{ab} + \epsilon^2 R^2_{ab} + ...) \end{pmatrix} \tag{2.47}$$

where R^2_{ab} is the first correction to the intrinsic curvature of the metric $g^0_{ab} + \epsilon^2 g^2_{ab}$. Its explicit form is

$$R^2_{ab} = \tfrac{1}{2} G_{;ab} + \tfrac{1}{4} G g^0_{ab}. \tag{2.48}$$

The Ricci curvature of the boundary metric is

$$R[h] = 6l^2 + l^2\epsilon^2 R^0 + 2l^2\epsilon^2 G + ... \tag{2.49}$$

Since the metric is degenerate *all* curvature invariants start at leading order. One way to construct the action would be to repeat the usual procedure and take a counterterms of the form

$$\mathcal{S}_{\rm ct} = \int d^5x\sqrt{h}\left(a + bR[h] + cR[h]^2 + dR_{ef}[h]R^{ef}[h] + ...\right) \tag{2.50}$$

where we choose the coefficients to cancel divergences. The key difference from (2.45) is that all terms will contribute to the cancel of the divergences at each order in ϵ and there is no "natural" choice of coefficients.

Perhaps a more natural way of removing the divergences would be to write the counterterm action in terms of not only the induced metric h but also the induced matter fields on the boundary. This is not necessary in this case but if there were other divergences arising from the bulk matter fields which could only be cancelled by including boundary matter fields in the counterterms it would become so. The induced matter fields on the boundary are the scalar field ϕ^b and potentials B_{ef} and B_e related to the bulk two-form potential as

$$A_{mn} = B_{mn} + B_{[m}n_{n]}, \tag{2.51}$$

where n is the unit normal to the boundary. With our previous gauge choice, the only relevant induced field is the two-form which satisfies

$$\begin{aligned} B_{\alpha\beta} &= -l^{-2}\epsilon^{-2} + \tfrac{1}{2}R^0 \ln\epsilon + ... \qquad \rightarrow \qquad H_{\alpha\beta e} = 0; \\ H &= 2l^{-2}\eta^0_3 + O(\epsilon^3)...., \end{aligned} \tag{2.52}$$

where H is the field strength of B. From this field strength we can construct the covariant object

$$H_{abc}H^{abc} = 24l^2(1 - G\rho^2). \tag{2.53}$$

Putting this together with (2.44) and (2.49) we find that we can express the counterterm action as

$$\mathcal{S}_{\rm ct} = \frac{l}{\kappa_6^2}\int d^5x\sqrt{h}\left(1 - (R[h] - \tfrac{1}{4}H^2)\ln\epsilon\right). \tag{2.54}$$

So it seems that we have effectively evaded the potential problems caused by the metric being degenerate in this case: we have written the counterterms in terms of seemingly covariant quantities on the boundary. However although we have used "covariant" quantities the action is not covariant: specific coordinate choices for the boundary are still necessary for it to be applicable as we will see.

Given a renormalised action we need to clarify the conditions under which this will indeed give a finite answer for solutions of the six dimensional field equations because hidden in our analysis are various assumptions, both obvious and subtle.

It is apparent that we cannot take an arbitrary six-dimensional solution and expect (2.54) to render the action finite. This manifestly will not work for the six-dimensional Schwarzschild solution, for example, even though this does satisfy the field equations. The counterterms will only work for solutions where the metric is asymptotic to $AdS_3 \times S^3$. The subtleties arise in defining whether a spacetime does need indeed asymptote to this.

There are several equivalent ways of stating the conditions under which a metric is asymptotically AdS. If one can find a coordinate system near the boundary such that the metric admits an expansion of the form (2.11), (2.12) then the spacetime must be asymptotically AdS. Such a condition is effectively a generalisation of the conditions on metric components given in in [25] for four dimensions and in [26] for three dimensions when g^0 is flat. One could also state the asymptotic conditions in terms of Penrose's definitions of conformal infinity [24]: if a spacetime of negative curvature has a regular conformal boundary then the spacetime is asymptotically AdS.

Let us try to define under what circumstances a solution of a particular set of six-dimensional field equations is asymptotically $AdS_3 \times S^3$. For the particular Lagrangian under consideration here we have effectively derived the radial dependence of metric components such that the metric is asymptotically of this form in a natural extension of the work of [25] and [26].

There is however an important and subtle caveat, related to the lack of covariance of a degenerate boundary. The divergences are covariant under coordinate transformations of the AdS_3 and the S^3 parts of the metric separately but are not covariant under the most general coordinate transformations which mix the two parts of the metric.

The simplest (and most physical) example of this is the following. In three dimensions we have a scalar matrix $\tilde{T}_{ij}$ which lies in the symmetric traceless representation of $SO(4)$ and corresponds to switching on $l = 2$ spherical harmonics on the three-sphere. From the six-dimensional perspective we can also switch on $l = 1$ harmonics on the three-sphere: this would correspond in three dimensions to a vector representation J_i of $SO(4)$. Now J_i is not part of the consistent truncation in three dimensions to supergravity [18] and is thus not included in our analysis. From the six-dimensional perspective, however, there is no reason not to switch on J_i. This corresponds to having a non-zero electric dipole moment of the F_3 charge distribution.

Let us clarify here what we mean by dipoles in this context. As usual the electric monopole moment of the charge distribution is given by the (coordinate invariant) expression

$$q = \int_{\Sigma_3} *F_3, \tag{2.55}$$

where Σ_3 is some appropriate closed 3-cycle. Higher pole moments of the charge distribution can be defined once we have fixed a coordinate choice. For a static charge distribution the 3-form will usually be defined in terms of a function which admits an expansion of the form

$$\phi = q_0\rho^2 + q_1 g_1(x^a)\rho^3 + q_2 g_2(x^a)\rho^4 + ... \tag{2.56}$$

where ρ has a first order zero on the conformal boundary and $g_i(x^a)$ are i-th harmonics on the sphere. Then q_i can, with suitable normalisation, be identified as the i-pole moment of the charge distribution, in direct analogy with classical electromagnetism.

Since the dipole moment is a coordinate dependent quantity it will always be possible to eliminate it by making an appropriate coordinate redefinition. Coordinate transformations can be used to eliminate all the odd powers of ρ in (2.56); this effectively removes J_i and brings us back into the coordinate system of Fefferman and Graham. Since the counterterms are not covariant we should not however exclude the possibility of new divergences/counterterms being needed when we are not in the special coordinate system where these moments vanish. If we are in the dipole coordinate frame, and regulate the boundary with the "natural" radial parameter, the renormalised action may no longer be finite.

Put another way, we have determined all possible divergences from the three-dimensional perspective. However when we uplift to six dimensions and do a coordinate transformation which, for example, induces a dipole moment new divergences may appear. In three dimensions the counterterm action renormalises the action for *all* solutions of the field equations. In six dimensions it does not because not all six-dimensional solutions solve the three-dimensional equations of motion: in general we will need to switch on massive excitations in three dimensions as well.

To show that there are such divergences in six dimensions let us consider an explicit solution which has a dipole moment, a distribution of dyonic black strings such that

$$\begin{aligned} ds_6^2 &= f^{-1}(-dt^2 + dx^2) + f(dr^2 + r^2 d\Omega_3^2); \\ F_3 &= df^{-1} \wedge dt \wedge dx + *(df^{-1} \wedge dt \wedge dx), \end{aligned} \tag{2.57}$$

where f is a harmonic function in flat space. An appropriate choice to ensure that the metric is asymptotically $AdS_3 \times S^3$ with the same curvature radius as before and a non-zero dipole moment is to take

$$f = \frac{1}{l^2 r^2}(1 + g(x^a) r^{-1}), \tag{2.58}$$

where g is an $l = 1$ harmonic on the three-sphere and hence satisfies $g_{;a}{}^{a} = -3g$. The spacetime is manifestly singular at $r = 0$ but this singularity does not affect our discussion of IR divergences. (It will of course be relevant for UV divergences.) The dipole moment of the charge distribution can manifestly be removed by a coordinate redefinition

$$\begin{aligned} \tilde{r} &= r - \tfrac{1}{2} g(x^a) + ..; \\ d\tilde{\Omega}_3^2 &= \left(1 + g(x^a) r^{-1} + ...\right) d\Omega_3^2, \end{aligned} \tag{2.59}$$

but the action divergences will not necessarily be preserved under such a coordinate transformation. That is, if we regulate the boundary at some fixed $\tilde{r} = \tilde{R}$, the renormalised action is guaranteed by our construction to be finite. However, if we regulate the boundary at some fixed $r = R$ it is no longer guaranteed to be finite. Note also that in removing the dipole moment we will in general induce a non-zero quadrupole moment; in other words we will not be able to eliminate the r^2 corrections to the metric.

The bulk part of the action is exactly zero in this case since the three form is self dual but the IR divergences from the boundary terms when we cut off at $r = R \gg 1$ are

$$\mathcal{S}_{\rm div} = -\frac{1}{\kappa_6^2} \int d^2x \int d^3x \sqrt{\eta^0} \left(R^2 - \tfrac{3}{2} l^{-1} g(x^a) R + ...\right), \tag{2.60}$$

where ellipses denote finite terms. There is indeed an additional divergence linear in R *but* this vanishes since the integral of any harmonic over the sphere vanishes. Evidently the next order terms will not be zero automatically since the integral of, for example, g^2 over the sphere does not vanish but such terms are finite in R. There are in principle contributions to the action from the inner boundary cutting off the singularity, but these are zero independently of how we cut off the singularity and so have no role here.

Now we need to evaluate the counterterm action on this solution. For the same reasons as above, the non-logarithmic counterterm does not have any divergent contributions dependent on g and is just given by

$$\mathcal{S}_{\rm ct} = \frac{1}{\kappa_6^2} \int d^2x \int d^3x \sqrt{\eta^0} \left(R^2 + ...\right). \tag{2.61}$$

However the logarithmic counterterm gives the following

$$\mathcal{S}_{\rm ct} = \frac{1}{l^2 \kappa_6^2} \int d^2x \int d^3x \sqrt{\eta^0} \left(\tfrac{105}{2} g(x^a)^2 + \tfrac{1}{2} (\partial g)^2\right) \ln R, \tag{2.62}$$

which does not vanish or cancel any other logarithmic divergence. We cannot just discard the logarithmic term, since it is necessary to cancel divergences for other solutions, so to cancel this divergence we would need to subtract another logarithmic term from the counterterm action. We have just shown explicitly that the counterterms are not covariant in six dimensions, even though the counterterms are written in terms of apparently covariant quantities in the boundary.

Now the question is: would it be possible to write down a (covariant) counterterm action which eliminates the divergences both for uplifted solutions from three dimensions *and* six-dimensional solutions which have a dipole moment. After all, as we pointed out earlier, the choice of counterterms was not unique. We chose to cancel logarithmic divergences with R and H^2 terms but we could have used, for example, R and R^2 instead.

It is in fact possible to fix the counterterms so that divergences are eliminated both when there is a dipole moment and when there is not. This however is beside the point since there are still an *infinite* number of coordinate transformations of the boundary that one could make in six dimensions for which the resulting counterterms would not remove divergences. For example, starting with pure $AdS_3 \times S^3$ in the Poincaré coordinate system, we could do a coordinate transformation of the form

$$\tilde{\rho} = \rho + a(x^a)_1 \rho^2 + a(x^2)_2 \rho^3 ..., \tag{2.63}$$

where $a(x^a)_i$ are arbitrary functions on the sphere. Just plugging this metric into the action, and regulating the boundary at some small fixed $\tilde{\rho}$ we will find new uncancelled divergences. We can adjust the counterterms so that the divergences cancel for some choices of $a(x^a)_i$ but they will never cancel for all choices. However hard we try we cannot get a covariant counterterm action.

It is useful to understand this from two slightly different but complimentary perspectives. Above we have been talking about bulk coordinate redefinitions: we change the bulk coordinates and then regulate the boundary with the new "natural" radial parameter. However, one could keep the bulk solution in the same coordinate system and just deform the boundary. The two perspectives are equivalent. From what we have said it is obvious that under such a boundary deformation the counterterm action will not in general still renormalise the action. To convince oneself of this, use an explicit solution, such as pure $AdS_3 \times S^3$ again, and cutoff the boundary in a non-standard way with some function $f(x^\alpha, \rho, x^a) \to 0$; there must in general be uncancelled divergences when we explicitly evaluate our "renormalised" action.

So given a generic metric which is asymptotically $AdS_3 \times S^3$ we cannot renormalise the action using (2.54) unless this solution is in the same class as the metric used to calculate (2.54). One way to define this is to demand that the metric admits a Kaluza-Klein reduction to three dimensions, using the ansatz (2.5), and explicitly identify the three-dimensional fields, order by order in ρ. It is easy to convince oneself that (2.57) does not admit such a reduction until one removes the dipole moment by a coordinate transformation.

In not every case will a coordinate redefinition be necessary to eliminate divergences. Where the metric corrections arise from gauge transformations or diffeomorphisms of three dimensional fields, the corrections are innocuous and will certainly not lead to additional divergences in the action. An explicit example is the following. Consider the six-dimensional metric

$$\begin{aligned} ds_6^2 &= \frac{l^2 r^2}{F}\left(-(1-\frac{2mF}{r^2})dt^2 + dy^2\right) + \frac{r^2}{l^2\left((r^2+a^2)(r^2+b^2)-2mr^2\right)}dr^2 \\ &\quad -2\left(b\cos^2\theta d\psi + a\sin^2\theta d\phi\right)dt - 2\left(a\cos^2\theta d\psi + b\sin^2\theta d\phi\right)dy \\ &\quad l^{-2}\left(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2\right); \\ F^{-1} &= \left(1 + \frac{a^2\cos^2\theta}{r^2} + \frac{b^2\sin^2\theta}{r^2}\right). \end{aligned} \tag{2.64}$$

This is a metric for (the near horizon geometry of) rotating dyonic strings and is a solution to the field equations, with appropriate choice of three-form. To use the counterterm action (2.54) we first need to eliminate the crossterms to sufficient order which in this case can be achieved simply by shifting the angular variables [27]

$$\begin{aligned} d\tilde{\psi} &= d\psi - l^2\left(bdt + ady\right); \\ d\tilde{\phi} &= d\phi - l^2\left(adt + bdy\right), \end{aligned} \tag{2.65}$$

so that the metric becomes

$$\begin{aligned} ds_6^2 &= -\frac{l^2\left((r^2+a^2)(r^2+b^2)-2mr^2\right)}{r^2}dt^2 + l^2r^2(dy - \frac{ab}{r^2}dt)^2 \\ &\quad + \frac{r^2}{l^2\left((r^2+a^2)(r^2+b^2)-2mr^2\right)}dr^2 + l^{-2}\left(d\theta^2 + \sin^2\theta d\tilde{\phi}^2 + \cos^2\theta d\tilde{\psi}^2\right), \end{aligned} \tag{2.66}$$

in which form it is apparent that the geometry is a direct product of two three-dimensional spaces. The coordinate transformation amounts to a trivial gauge transformation of Abelian gauge fields in three dimensions.

All of the above has focussed on the explicit metric dependence of the counterterm action. This is not the only way in which the six-dimensional action is not covariant: it manifestly also depends on the explicit gauge choice for the two-form potential. This follows from (2.36), (2.51) and (2.54): one can find gauge transformations of the bulk two-form potential which do not leave the divergent part of the action invariant. Again this is related to the lack of full six-dimensional covariance. So not only do we have to bring the metric into the prescribed form but also we have to bring the two form potential into the same gauge as chosen above.

The conclusion of our analysis is the following. Given an asymptotically $AdS_3 \times S^3$ solution of the six-dimensional field equations we can renormalise the action provided that the coordinate frame is such that all fields can be matched directly to the perturbative expansions of the three-dimensional fields given here and we regulate the boundary suitably. The most obvious regulation is $\rho = \epsilon$ but other regulations are allowed provided that the "crossterms" between AdS_3 and S^3 coordinates are sufficiently subleading. Three-dimensional covariance means that we can certainly also regulate with any function of AdS_3 coordinates which has a first order zero on the (degenerate) conformal boundary.

If the bulk fields cannot be directly matched we should change coordinates before evaluating the counterterms, and again cutoff the boundary suitably. Equivalently we could cutoff the boundary not with the natural radial parameter but in such a way as to be equivalent to a regulated boundary in our preferred coordinate frame. In the next section we will give an explicit demonstration of how this process works. Thus although we would have hoped to express everything covariantly from a six-dimensional perspective we are forced back to three dimensions if we want covariance of the counterterm action. Explicit calculations are more naturally carried out in the lower dimension, although the higher dimensional procedure can be made consistent and well-defined.

3 Divergences and counterterms for $AdS_5 \times S^5$ spacetimes

Of greater interest than the toy example discussed in the previous section are the compactifications of eleven dimensional supergravity on S^4 and S^7 and of type IIB supergravity on S^5. In each case one can truncate the higher dimensional Lagrangian to the metric and p-form and show that there is a consistent spherical reduction retaining only the massless modes [28], [29]. So we could proceed as before, solving for all the lower dimensional divergences (which will not just be purely gravitational) and then uplifting to derive the higher dimensional counterterms. However the issues and analysis would simply repeat the previous section: we will have the same lack of covariance and be forced to choose a particular gauge for the p-form and a particular coordinate frame (and/or regulation) in order to apply our higher-dimensional counterterm action.

In this section we will focus instead on answering a question that motivates trying to renormalise the action directly in higher dimensions: given an arbitrary distribution of

D3-branes

$$\begin{aligned} ds^2_{10} &= \frac{1}{\sqrt{D}} dx \cdot dx_4 + \sqrt{D} dy \cdot dy_6; \\ F_5 &= dD^{-1} \wedge dx_4 + (*_6 \; dD), \end{aligned} \tag{3.1}$$

where D is an harmonic function on R^6, can we calculate the renormalised action? This D3-brane distribution is a solution of the equations of motion for type IIB supergravity truncated to the metric and self-dual 5-form F_5 with action

$$\mathcal{S}_{10} = \frac{1}{2\kappa^2_{10}} \int d^{10}x \sqrt{g} \left(R - \frac{1}{4 \cdot 5!} F_5^2 \right) - \frac{1}{\kappa^2_{10}} \int d^9 x K \sqrt{h}, \tag{3.2}$$

where as usual we impose by hand the self-duality condition. As before to reproduce the lower dimensional action upon Kaluza-Klein reduction we need to work in the canonical ensemble and add a boundary term

$$\delta \mathcal{S}_{10} = \frac{1}{8 \cdot 5! \kappa^2_{10}} \int d\Sigma^{\sigma} A^{\mu\nu\rho\tau} F_{\sigma\mu\nu\rho\tau}. \tag{3.3}$$

From supersymmetry the action should be zero for any distribution of positive tension branes since the renormalised action is zero when all the branes are located at the same point and separating the branes preserves supersymmetry. The goal of this section is to prove this.

Ten-dimensional fields can be reduced to five dimensions using a Kaluza-Klein ansatz analogous to (2.5):

$$\begin{aligned} ds^2_{10} &= \Delta^{\frac{1}{2}} ds^2_5 + l^{-2} \Delta^{-\frac{1}{2}} T^{-1}_{ij} d\mu^i d\mu^j; \\ F_5 &= G_5 + *G_5; \\ G_5 &= -lU\epsilon_5 + l^{-1} (T^{-1}_{ij} * dT_{jk}) \wedge (\mu^k D\mu^i); \\ *G_5 &= \frac{1}{5!} \epsilon_{i_1..i_6} \big(l^{-4} U \Delta^{-2} D\mu^{i_1} \wedge ... \wedge D\mu^{i_5} \mu^{i_6} \\ &\quad -5 l^{-4} \Delta^{-2} D\mu^{i_1} \wedge .... \wedge D\mu^{i_4} \wedge DT_{i_5 j} T_{i_6 k} \mu^j \mu^k \Big), \end{aligned} \tag{3.4}$$

where as in the previous section

$$U = 2 T_{ij} T_{jk} \mu^i \mu^k - \Delta T_{ii}, \qquad \Delta = T_{ij} \mu^i \mu^j, \qquad \mu^i \mu^i = 1. \tag{3.5}$$

Here the indices i, j are $SO(6)$ indices and T_{ij} is a symmetric unimodular tensor. We have truncated the $SO(6)$ Yang-Mills gauge fields to zero; this is consistent with the field equations. Truncating the vectors is permissible since the analysis will never be covariant from the ten-dimensional perspective; we will always have to choose a particular coordinate frame. Truncating the vectors simply restricts this frame to be one in which all cross-terms vanish and restricts further the class of ten-dimensional solutions to which we can apply the renormalised action.

The equations of motion can be derived from the five-dimensional Lagrangian

$$\mathcal{L}_5 = R - \tfrac{1}{4}T^{-1}_{ij}(\partial T_{jk})T^{-1}_{kl}(\partial T_{li}) - \tfrac{1}{2}l^2\left(2T_{ij}T_{ij} - (T_{ii})^2\right). \tag{3.6}$$

Expanding the fields in the usual way and solving the equations of motion we find that

$$\begin{aligned}
T_{ij} &= \delta_{ij} + \rho^2 t_{ij} + \rho^2\ln\rho\tilde{t}_{ij} + \rho^4 T^0_{ij} + \rho^4\ln\rho T^{(1)}_{ij} + \rho^4(\ln\rho)^2 T^{(2)}_{ij}; \\
T^{(0)}_{ii} &= \tfrac{1}{2}t_{ij}t_{ji}, \quad T^{(1)}_{ii} = t_{ij}\tilde{t}_{ji}, \quad T^{(2)}_{ii} = \tfrac{1}{2}\tilde{t}_{ij}\tilde{t}_{ji}; \\
\mathrm{tr}((g^0)^{-1}g^4) &= -\tfrac{1}{48}\tilde{t}_{ij}\tilde{t}_{ji} - \tfrac{1}{6}t_{ij}t_{ji}; \\
\mathrm{tr}((g^0)^{-1}h^{(1)}) &= -\tfrac{1}{3}t_{ij}\tilde{t}_{ji}; \quad \mathrm{tr}((g^0)^{-1}h^{(2)}) = -\tfrac{1}{6}\tilde{t}_{ij}\tilde{t}_{ji},
\end{aligned} \tag{3.7}$$

where we have assumed g^0 is flat since this is the case of interest here. Here t_{ij} corresponds to a vacuum expectation value for a dual scalar operator of dimension two whilst $\tilde{t}_{ij}$ corresponds to a source for such an operator.

This is all we need to determine the action divergences, although the field equations do fix the $T^{(i)}_{ij}$ entirely (although as usual only the trace of g^4 is determined [9]). Explicitly calculating the IR divergences in the action we find

$$\mathcal{S}_{\mathrm{div}} = -\frac{1}{l^3\kappa_5^2}\int d^4x\sqrt{g^0}\left(3\epsilon^{-4} + \tfrac{1}{8}\tilde{t}_{ij}\tilde{t}_{ji}\ln\epsilon\right). \tag{3.8}$$

Note also that there are no finite terms in the boundary action when g^0 is flat. The divergences can be cancelled by the covariant boundary action

$$\mathcal{S}_{\mathrm{ct}} = \frac{l}{\kappa_5^2}\int d^4x\sqrt{h}\left(3 + \tfrac{1}{8}(T_{ii}-6) + \tfrac{1}{8}(T_{ii}-6)\ln\epsilon\right), \tag{3.9}$$

where T_{ij} is the induced scalar matrix on the boundary. For the specific Coulomb branch flow considered in [11], this reduces to the counterterm action given there.

These counterterms can be used to show that any BPS domain wall solution which uplifts to a distribution of D3-branes with zero dipole moment, has zero action but to do so we must address two subtleties. The counterterm action removes all IR divergences for such solutions - but there could still be finite contributions to the action from both the boundary at infinity and the interior, usually singular, boundary which need to be cancelled by additional finite counterterms.

However, we can show that there are no finite terms on the boundary at infinity. We said above that there are no finite terms on the boundary from the bulk action (3.8). There are also no finite terms induced by the counterterm action, when we evaluate it with the fields (3.7). From the arguments of the previous section any brane solution will have to be brought into this coordinate frame to use the (uplifted) renormalised action. There will be no finite terms from the IR boundary for any BPS brane solution since it can be expressed exactly in the form (3.7). In what follows we will show - from the ten-dimensional perspective - that there are no finite contributions from the interior. So (3.9) will ensure that the action for all such BPS brane distributions is zero, as required by supersymmetry.

Given the asymptotic expansions of the five-dimensional fields we can use the Kaluza-Klein ansatz to uplift to ten dimensions. This results in the following expansion for the metric

$$\begin{aligned} ds^2_{10} &= \left(1 + \tfrac{1}{2}\rho^2 G + \tfrac{1}{2}\rho^2 \ln\rho \tilde{G} + ..\right)\left[\frac{d\rho^2}{\rho^2} + (\frac{1}{\rho^2} + ..)dx^\alpha dx_\alpha\right] \\ &\quad + \left(1 - \tfrac{1}{2}\rho^2 G - \tfrac{1}{2}\rho^2 \ln\rho \tilde{G}\right)\left(g^0_{ab} + \rho^2 g^2_{ab} + \rho^2 \ln\rho \tilde{g}^2_{ab}\right) dx^a dx^b; \\ g^2_{ab} &= -\frac{1}{2}G_{;ab} - G g^0_{ab}; \quad \tilde{g}^2_{ab} = -\frac{1}{2}\tilde{G}_{;ab} - \tilde{G} g^0_{ab}, \end{aligned} \tag{3.10}$$

where both G and $\tilde{G}$ are $l = 2$ harmonics on the five-sphere and are related to the five-dimensional scalar matrices as

$$G = t_{ij}\mu^i\mu^j; \quad \tilde{G} = \tilde{t}_{ij}\mu^i\mu^j. \tag{3.11}$$

By uplifting the fields we could formally write the counterterm action in ten dimensions, to be used with the caveat that the ten-dimensional fields must first be brought into the same coordinate frame as in (3.10). We would also have to fix a gauge for the 4-form and bring any solution into that same gauge and use the radial cutoff $\rho = \epsilon$ to regulate the boundary.

Now we would like to use the five-dimensional counterterm action to renormalise the action for a particular distribution of D3-branes, namely two separated stacks of branes. This will provide an explicit example of how the covariant five-dimensional counterterm action can be used to renormalise the ten-dimensional action. Such a distribution is described by a harmonic function of the form

$$D \propto \left(\frac{p}{|\mathbf{r} - \mathbf{a}|^4} + \frac{q}{|\mathbf{r} - \mathbf{b}|^4}\right), \tag{3.12}$$

corresponding to p D3-branes placed at $\mathbf{r} = \mathbf{a}$ and q D3-branes placed at $\mathbf{r} = \mathbf{b}$. By shifting the coordinate system we can eliminate the dipole moment of the charge distribution and bring the harmonic function into the form

$$D = \frac{1}{2l^4}\left(\frac{1}{|\mathbf{r} - \mathbf{a}|^4} + \frac{1}{|\mathbf{r} + \mathbf{a}|^4}\right);$$

$$= \frac{1}{l^4}\left(\frac{1}{r^4} + \frac{a^2}{r^6}(\cos^2\tilde{\theta} - \tfrac{1}{6}) + ...\right), \qquad (3.13)$$

where for simplicity we have also assumed that the two stacks are equal in number and in the second line we have used explicit coordinates on R^6. Let us now evaluate the action for such a distribution. There are three contributions to (3.2) and (3.3), from boundaries at the brane locations and from the boundary at infinity.

We deal with the latter first. The metric defined by (3.1) and (3.13) can be brought into the form (3.10) by coordinate transformations of the form

$$\begin{aligned} r &= \rho^{-1} + \frac{a^2}{6}(\cos^2\theta - \tfrac{1}{6})\rho + .. \\ \tilde{\theta} &= \theta + \frac{a^2}{3}\cos\theta\sin\theta\rho^2 + ..., \end{aligned} \qquad (3.14)$$

where

$$G = \frac{a^2}{3}(\cos^2\theta - \tfrac{1}{6}), \qquad (3.15)$$

which is manifestly an $l = 2$ harmonic on the five-sphere. This function corresponds to a five-dimensional scalar matrix of the form

$$t_{11} = \tfrac{5}{18}a^2, \quad t_{ij} = -\tfrac{1}{18}a^2\delta_{ij} \quad \forall i \neq 1, \qquad (3.16)$$

with an appropriate choice of μ^i, which is manifestly traceless as required. Having matched the ten-dimensional fields to the asymptotic expansion of the five-dimensional fields, we can determine the IR boundary contribution to the action using the five-dimensional action and counterterms already determined. This vanishes as we previously claimed; there is no finite contribution.

All that remains is to show that there is no contribution to the action from the interior (generically singular) boundaries. (A generic discrete distribution of branes will be singular at (all but one of) the brane locations [30].) At each brane location we need to evaluate the term

$$\delta\mathcal{S}_{10} = \frac{1}{8\cdot 5!\kappa_{10}^2}\int d\Sigma^\sigma A^{\mu\nu\rho\tau}F_{\sigma\mu\nu\rho\tau} \propto \int (n\cdot\partial(\ln D))d\Sigma, \qquad (3.17)$$

where n is the normal to the boundary. Since there are by definition source terms for the harmonic function at the boundary the integrand will always be singular here. However the boundary must always have zero nine-volume and so the integral vanishes.

Consider a general (continuous) distribution of D3-branes in R^6. Then (3.17) is defined by choosing a 5-surface which completely encloses the distribution. The integral must be independent of deformations of the 5-surface under which the distribution is still enclosed. Therefore we can always choose the 5-surface to be the outer boundary of the distribution itself. For example, suppose we have a ball distribution of branes in R^2 contained within $x^2 + y^2 \leq a^2$: then we can choose the 5-surface to be $x^2 + y^2 = a^2$ times the origin in the remaining R^4. Now the 5-volume of this surface manifestly vanishes and so thus will the integral (3.17) even though D is singular here.

The only way we could get a finite volume 5-surface would be to arrange the branes within a finite sized 5-ball in R^6. However, this geometry is unphysical since it involves "branes" of negative charge and tension which are neither D3-branes nor anti-D3-branes [31]. So we can exclude this geometry from our analysis. For all other continuous or discrete D3-brane distributions there can be no contribution from inner boundary terms. Of course similar arguments are used to show that there are no contributions from Gibbons-Hawking boundary terms either. This completes the proof that the renormalised action is zero for all such BPS D3-brane distributions.

The procedure for renormalising the ten-dimensional action outlined above could be extended systematically to the most general asymptotically $AdS_5 \times S^5$ solutions. In principle one could derive all the counterterms needed for $SO(6)$ gauged supergravity in five dimensions. One could then use the Kaluza-Klein ansatz to uplift the solution. By bringing a given ten-dimensional solution into this same form by coordinate transformations, order by order in the expansion near infinity, we could identify the effective five-dimensional field expansions corresponding to the ten-dimensional solution and then use the *covariant* five-dimensional counterterm action to remove the IR divergences.

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